Gaussian Elimination

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August 7, 2025 and latest revision September 25, 2025

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`Sir Isaac Newton: If I have seen further, it is by standing on the shoulders of giants.`

1 Introduction to Gaussian Elimination

The central problem of linear algebra is *solving linear equations*. And the good practical method to find solutions to a system of linear equations is Gaussian elimination (also called Gauss elimination or Gauss reduction). In this method, we systematically proceed to reduce the coefficient matrix A of a system of linear equations $Ax = b$ to an equivalent triangular or echelon form using elementary row operations. In the examples that follow in the article, we shall demonstrate the algorithm to solve systems consisting of few equations with few variables; however, the algorithm is constantly used to solve large systems containing more than thousand equations and variables. Though the Gauss elimination solves the linear equations, we need to keep in mind the geometric picture involved: the *row picture*— intersection of planes, and the *column picture*— combination of column vectors.

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Example 1 : **Gauss Elimination when Exact Solution Exists (with no row interchange)**
Solve the following system of linear equations.

$$\begin{aligned} L_1: \quad & 2x_1 + x_2 + x_3 = 5 \\ L_2: \quad & 4x_1 - 6x_2 = -2 \\ L_3: \quad & -2x_1 + 7x_2 + 2x_3 = 9 \end{aligned}$$

`PART A: Using elementary row operations on equations.`

In the first equation L_1, the coefficient of x_1, which is 2, is the *pivot*. Using the pivot 2 in the first equation, we eliminate x_1 from the second and third equations. Replace L_2 by L_2' with $L_2' = -2L_1 + L_2$ and replace L_3 by L_3' with $L_3' = L_1 + L_3$. We obtain the following equivalent system.

$$\begin{aligned} L_1: \quad & 2x_1 + x_2 + x_3 = 5 \\ L_2': \quad & -8x_2 - 2x_3 = -12 \\ L_3': \quad & +8x_2 + 3x_3 = 14 \end{aligned}$$

Using the pivot -8 in the second equation, we eliminate x_2 from the third equation. Replace L_3' by L_3'' with $L_3'' = L_2' + L_3'$. We obtain the following triangular system.

$$\begin{aligned} L_1: \quad & 2x_1 + x_2 + x_3 = 5 \\ L_2': \quad & -8x_2 - 2x_3 = -12 \\ L_3'': \quad & x_3 = 2 \end{aligned}$$

Forward elimination has produced the pivots 2, -8, 1. This last system is solved backwards, bottom to top. The last equation gives $x_3 = 2$. Substituting in second equation, we get $x_2 = 1$. Then the first equation gives $x_1 = 1$. This procedure is called *back-substitution*.

The given linear system has a unique solution $(x_1, x_2, x_3) = (1, 1, 2)$. Geometrical description (row picture): Here the three planes intersect in exactly one point (*no free variable*).

`PART B: Using matrix algebra.`

We proceed to solve the same problem using elementary row matrices. The linear system is arranged in matrix form $Ax = b$.

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

Using $a_{11} = 2$ as the first pivot, we use elimination to have 0's below it in the first column of A.

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 14 \end{bmatrix}$$

Using $a_{22} = -8$ as the second pivot, we use elimination to have 0 below it in the second column of A.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -12 \\ 14 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 2 \end{bmatrix}$$

The entree $a_{33} = 1$ is the third pivot. Back-substitution gives the solution vector $(x_1, x_2, x_3) = (1, 1, 2)$.

`PART C: Reducing augmented matrix to row echelon form.`

A system of linear equations can be solved by working with its augmented matrix, say, M, where $M = [A, b]$, and reducing it to row echelon matrix.

$$M = \begin{bmatrix} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix}$$

We proceed as in `PART B` to obtain following row equivalent matrices.

$$\begin{bmatrix} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

We can write the echelon matrix as linear equations, and applying back-substitution to these equations yields $x_1 = 1$, $x_2 = 1$, $x_3 = 2$.

`Row Canonical Form`
Row echelon matrix can be reduced to *row canonical form* with the properties: (i) each pivot is equal to 1 and (ii) each pivot is the only nonzero entry in its column.

For the present example, the row echelon matrix, with circles around the pivots, is shown below.

$$\begin{bmatrix} \textcircled{2} & 1 & 1 & 5 \\ 0 & \textcircled{-8} & -2 & -12 \\ 0 & 0 & \textcircled{1} & 2 \end{bmatrix}$$

We now proceed to reduce this matrix to the equivalent row canonical form. Pivot in the last row is 1, and it lies in the third column. To introduce zeros in the third column above this pivot, we apply the following elementary row operations: Replace R_2 by $R_2 + 2R_3$, and replace R_1 by $R_1 - R_3$.

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 3 \\ 0 & -8 & 0 & -8 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

To change pivot -8 of row 2 to 1 in the product matrix, we multiply row 2 by $-1/8$, that is, replace R_2 by $-R_2/8$.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1/8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 3 \\ 0 & -8 & 0 & -8 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

To introduce zero in the second column above the pivot 1 in row 2 of the product matrix, we apply the following elementary row operation: Replace R_1 by $R_1 - R_2$.

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

To change pivot 2 in the first row of the product matrix to 1, we divide the first row by 2, that is, replace R_1 by $R_1/2$.

$$\begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The product matrix is now in row canonical form and we directly get the solution: $x_1 = 1$, $x_2 = 1$, $x_3 = 2$.

LU decomposition for coefficient matrix A in *Example 1*

Since the system of linear equations in *Example 1* has an exact solution, the coefficient matrix A is non-singular. Matrix A is reduced to upper triangular form U as shown below.

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

The elimination or elementary matrices E_1 and E_2, in that order, during Gaussian reduction are given below.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

We have the following relations.

$$E_2E_1A = U$$

$$(E_2E_1)^{-1}E_2E_1A = (E_2E_1)^{-1}U$$

$$A = E_1^{-1}E_2^{-1}U$$

The lower triangular matrix L is equal to $E_1^{-1}E_2^{-1}$. The inverse matrices E_1^{-1} and E_2^{-1} are found by inspection.

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

We can check $LU = A$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} \overset{\checkmark}{=} \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

Triangular factorization $A = LDU$

The matrix U in LU decomposition can be easily factored into diagonal matrix D of pivots and matrix U having 1s on the diagonal. For the present *Example 1*, triangular factorization $A = LDU$ is shown below.

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{8} \\ 0 & 0 & 1 \end{bmatrix}$$

There is *some* freedom in the order in which elimination steps are undertaken in Gaussian elimination; but, there is *no freedom* in the final L, D and U those are obtained. The LDU factorization and LU factorization are uniquely determined by A.

2 Row Equivalence, Rank of a Matrix

A matrix A is said to be *row equivalent* to matrix B, written $A \sim B$, if B can be obtained from A by a sequence of elementary row operations. Every matrix A can be reduced to an row equivalent echelon matrix; the echelon matrix can be further reduced to a unique matrix in row canonical form.

The *rank* of a matrix A, written $\text{rank}(A)$, is equal to the number of pivots in an echelon form of A. The rank of all equivalent matrices is the same.

The rank is a very important property of a matrix and, depending on the context in which the matrix is used, it will be defined in many different ways. Of course, all the definitions lead to the same number.

In *Example 1*, the coefficient matrix A and the augmented matrix M are as given below.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}, \quad M = \begin{bmatrix} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix}$$

These matrices are reduced to their equivalent echelon forms, say A'' and M''.

$$A'' = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}, \quad M'' = \begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The rank of matrix A and the rank of matrix M are the same, that is equal to 3.

We arrive at the following theorem.

A system of linear equations in n unknowns with augmented matrix $M = [A, b]$ has unique solution if and only if $\text{rank}(A) = \text{rank}(M) = n$.

3 Gaussian Elimination if Infinitely Many Solutions Exist

We now illustrate with an example the Gauss elimination applied to a system of linear equations having infinitely many solutions.

Example 2 :
Solve the system of three linear equations in four unknowns.

$$\begin{aligned} 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 &= 8.0 \\ 0.6x_1 + 1.5x_2 + 1.5x_3 - 5.4x_4 &= 2.7 \\ 1.2x_1 - 0.3x_2 - 0.3x_3 + 2.4x_4 &= 2.1 \end{aligned}$$

We will work with augmented matrix as shown below.

$$\left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{array}\right]$$

We eliminate x_1 from the second and third equations (rows) by replacing R_2 by $R_2 - 0.2R_1$ and R_3 by $R_3 - 0.4R_1$ to get the following equivalent matrix.

$$\left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & -1.1 & -1.1 & 4.4 & -1.1 \end{array}\right]$$

We eliminate x_2 from third equation by replacing R_3 by $R_3 + R_2$ to get the following equivalent matrix in echelon form.

$$\left[\begin{array}{cccc|c} \textcircled{3.0} & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & \textcircled{1.1} & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The pivots are circled. The coefficient matrix and the augmented matrix have the same rank, namely, 2. The variables x_1 and x_2 are the pivot variables, and the variables x_3 and x_4 are the free variables. The pivot variables can be expressed in terms of the free variables, and the free variables can be assigned arbitrary values. Now, from the second equation in echelon form, we have $x_2 = 1 - x_3 + 4x_4$. From this and the first equation, we have $x_1 = 2 - x_4$. Since x_3 and x_4 remain arbitrary, we have infinitely many solutions; if we choose a value of x_3 and a value of x_4, then the corresponding values of x_1 and x_2 are uniquely determined. Geometrical description (row picture): Here the three hyperplanes intersect in a plane (*two free variables*).

From *Examples 1 and 2*, we arrive at the following theorem.
Consider a system of linear equations in n unknowns with augmented matrix $M = [A, b]$. Then
(a) The system has a solution if and only if $\text{rank}(A) = \text{rank}(M)$.
(b) The solution is unique if and only if $\text{rank}(A) = \text{rank}(M) = n$.
(c) The system has infinitely many solutions if $\text{rank}(A) = \text{rank}(M) < n$.

4 Gaussian Elimination if no Solution Exists

The following example illustrates the Gauss elimination applied to a system of linear equations having no solution.

Example 3 :
Solve the following system of linear equations.

$$\begin{aligned} x_1 + x_2 - 2x_3 + 3x_4 &= 4 \\ 2x_1 + 3x_2 + 3x_3 - x_4 &= 3 \\ 5x_1 + 7x_2 + 4x_3 + x_4 &= 5 \end{aligned}$$

We write the system as augmented matrix M and reduce it to echelon form as follows:

$$\left[\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 4 \\ 2 & 3 & 3 & -1 & 3 \\ 5 & 7 & 4 & 1 & 5 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 4 \\ 0 & 1 & 7 & -7 & -5 \\ 0 & 2 & 14 & -14 & -15 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 4 \\ 0 & 1 & 7 & -7 & -5 \\ 0 & 0 & 0 & 0 & -5 \end{array}\right]$$

The third row of the echelon matrix corresponds to the degenerate equation

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = -5$$

which has no solution. Thus the system has no solution. Note that the rank of the coefficient matrix is 2 and the rank of the augmented matrix is 3 and the two ranks are not equal.

Geometrical description (row picture): Here the three hyperplanes may intersect pairwise but have no common intersection.

From *Example 3*, we arrive at the following theorem.
Consider a system of linear equations in n unknowns with augmented matrix $M = [A, b]$. Then the system has no solution if $\text{rank}(A) \neq \text{rank}(M)$.

5 Gauss-Jordan Elimination Algorithm

The Gauss-Jordan elimination algorithm is similar in some ways to the Gaussian elimination algorithm, except that here each pivot is used to place 0's both below and above the pivot, not just below the pivot, before working with the next pivot. Also, one variation of the algorithm first *normalizes* each row, that is obtains a unit pivot, before it is used to produce 0's in the other rows, rather than normalizing the rows at the end of the algorithm.

Example 4 :
Let $A = \begin{bmatrix} 1 & -2 & 3 & 1 & 2 \\ 1 & 1 & 4 & -1 & 3 \\ 2 & 5 & 9 & -2 & 8 \end{bmatrix}$. Use Gauss-Jordan to find the row canonical form of A.

Use a_{11} as a pivot to obtain 0's below a_{11} by applying the operations "Replace R_2 by $-R_1 + R_2$" and "Replace R_3 by $-2R_1 + R_3$". This yields

$$A \sim \begin{bmatrix} 1 & -2 & 3 & 1 & 2 \\ 0 & 3 & 1 & -2 & 1 \\ 0 & 9 & 3 & -4 & 4 \end{bmatrix}$$

Multiply R_2 by $\frac{1}{3}$ to make the pivot $a_{22} = 1$, and then produce 0's below and above a_{22} by applying the operations "Replace R_3 by $-9R_2 + R_3$" and "Replace R_1 by $2R_2 + R_1$". These operations yield

$$A \sim \begin{bmatrix} 1 & -2 & 3 & 1 & 2 \\ 0 & 1 & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ 0 & 9 & 3 & -4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{11}{3} & -\frac{1}{3} & \frac{8}{3} \\ 0 & 1 & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 2 & 1 \end{bmatrix}$$

6 Finding Inverse of a Matrix

Let $Ax = b$ be a system of n linear equations with n unknowns. We can write this as $Ax = Ib$ where I is an $n \times n$ unit matrix. We now proceed to carry out elementary row operations on A (and therefore the same operations will apply to I) in order to attempt to reduce A to I. If we succeed then we get $Ix = A^{-1}b$. This means A is an invertible matrix and the linear system has a unique solution. Otherwise, in the reduction of matrix A, we get a row of 0's indicating that A is not invertible and the system has no solution.

In practice we solve problems by forming an $n \times 2n$ (block) matrix $M = [A, I]$. We row reduce M to echelon form. If the process generates a zero row in the A half of M, then A has no inverse. Otherwise A half is in triangular form, and has an inverse. Then we further row reduce M to its row canonical form $M \sim [I, A^{-1}]$.

Example 5 :
Find the inverse of the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$.

First we form the (block) matrix $M = [A, I]$ and row reduce M to an echelon form:

$$M = \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{array}\right]$$

In echelon form, the left half of M is in triangular form; hence A has an inverse. Next we further row reduce M to its row canonical form.

$$M \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & -1 & 0 & 4 & 0 & -1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right]$$

The identity matrix is now in the left half of the final matrix; hence the right half is A^{-1}. In other words,

$$A^{-1} = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}$$

Example 6 :
Find the inverse of $B = \begin{bmatrix} 1 & 3 & -4 \\ 1 & 5 & -1 \\ 3 & 13 & -6 \end{bmatrix}$.

We form the matrix $M = [B, I]$ and row reduce M to echelon form:

$$M = \left[\begin{array}{ccc|ccc} 1 & 3 & -4 & 1 & 0 & 0 \\ 1 & 5 & -1 & 0 & 1 & 0 \\ 3 & 13 & -6 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 2 & 3 & -1 & 1 & 0 \\ 0 & 4 & 6 & -3 & 0 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 2 & 3 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -2 & 1 \end{array}\right]$$

In row echelon form, M has a zero row in its left half; that is, B is not row reducible to a triangular form. Accordingly, B has no inverse.

7 Systems of Linear Equations and Linear Combinations of Vectors

The general system of linear equations

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ &a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{aligned} \tag{1}$$

may be rewritten as the following vector equation:

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \cdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \cdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \cdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_m \end{bmatrix} \tag{2}$$

This is the column picture, that is, the left hand side is the combination of the columns of the coefficient matrix.

In linear algebra, we have vectors whose elements come from the field of real numbers R, or field of complex numbers C; many of our operations also apply to vectors whose entries come from some arbitrary field K.

A vector v in K^n is said to be a *linear combination* of vectors $u_1, u_2, \ldots, u_m$ in K^n if there exist scalars $a_1, a_2, \ldots, a_m$ in K such that

$v = a_1u_1 + a_2u_2 + \cdots + a_mu_m$

Accordingly, the general system (1) of linear equations and the equivalent vector equation (2) have a solution if and only if the column vector of constants is a linear combination of the columns of the coefficient matrix. We can state this observation formally as a theorem.

A system $Ax = b$ of linear equations has a solution if and only if b is a linear combination of the columns of the coefficient matrix A.

Thus the answer to the problem of expressing a given vector v in K^n as a linear combination of vectors $u_1, u_2, \ldots, u_m$ in K^n reduces to solving a system of linear equations.

We write a column vector in two different ways: this is shown for a vector u below.

$$u = \begin{bmatrix} 7 \\ 34 \\ 5 \\ 18 \end{bmatrix}, \quad u = (7,\ 34,\ 5,\ 18)$$

Example 7 :
Write v as a linear combination of u_1, u_2, u_3, where $v = (3, 10, 7)$ and $u_1 = (1, 3, -2)$, $u_2 = (1, 4, 2)$, $u_3 = (2, 8, 1)$.

We can approach to solve the problem in two ways. Find the equivalent system of linear equations by writing $v = xu_1 + yu_2 + zu_3$. Alternatively, use the augmented matrix M of the equivalent system, where $M = [u_1, u_2, u_3, v]$. (Here u_1, u_2, u_3, v are the columns of M.)

The vector equation $v = xu_1 + yu_2 + zu_3$ for the given vectors is as follows:

$$\begin{bmatrix} 3 \\ 10 \\ 7 \end{bmatrix} = x \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + y \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + z \begin{bmatrix} 2 \\ 8 \\ 1 \end{bmatrix} = \begin{bmatrix} x + y + 2z \\ 3x + 3y + 8z \\ -2x + 2y + z \end{bmatrix}$$

We can write equivalent system of linear equations and then reduce the system to echelon form:

$$\begin{array}{lll} x + y + 2z = 3 & x + y + 2z = 3 & x + y + 2z = 3 \\ 3x + 3y + 8z = 10 & y + 2z = 1 & y + 2z = 1 \\ -2x + 2y + z = 7 & 4y + 5z = 13 & -3z = 9 \end{array}$$

The system is in triangular form. Back-substitution yields the unique solution $x = 2$, $y = 7$, $z = -3$. Thus $v = 2u_1 + 7u_2 - 3u_3$.

Alternatively, we form the augmented matrix $M = [u_1, u_2, u_3, v]$ of the equivalent system, and reduce M to echelon form:

$$M = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 3 & 3 & 8 & 10 \\ -2 & 2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 4 & 5 & 13 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -3 & 9 \end{bmatrix}$$

The last matrix corresponds to a triangular system that has a unique solution. Back-substitution yields the $x = 2$, $y = 7$, $z = -3$. Thus $v = 2u_1 + 7u_2 - 3u_3$.

Example 8 :
Write v as a linear combination of u_1, u_2, u_3, where $v = (2, 7, 10)$ and $u_1 = (1, 2, 3)$, $u_2 = (1, 3, 5)$, $u_3 = (1, 5, 9)$.

We form the augmented matrix $M = [u_1, u_2, u_3, v]$ of the equivalent system, and reduce M to echelon form:

$$M = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 5 & 7 \\ 3 & 5 & 9 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & 3 \\ 0 & 2 & 6 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

The third row corresponds to the degenerate equation $0x + 0y + 0z = -2$, which has no solution. Thus the system also has no solution, and v cannot be written as a linear combination of u_1, u_2, u_3.

Example 9 :
Write v as a linear combination of u_1, u_2, u_3, where $v = (1, 5, 4)$ and $u_1 = (1, 3, -2)$, $u_2 = (2, 7, -1)$, $u_3 = (1, 6, 7)$.

We write the augmented matrix $M = [u_1, u_2, u_3, v]$ of the equivalent system, and reduce M to echelon form:

$$M = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 3 & 7 & 6 & 5 \\ -2 & -1 & 7 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 3 & 9 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The last matrix corresponds to the following system with free variable z:

$$x + 2y + z = 1$$
$$y + 3z = 2$$

Thus v can be written as a linear combination of u_1, u_2, u_3 in many ways. For example, let the free variable $z = 1$, and by back-substitution, we get $y = -2$ and $x = 2$. Thus $v = 2u_1 - 2u_2 + u_3$.

`Remark:` Expressing a given vector v in K^n as a linear combination of vectors $u_1, u_2, \ldots, u_m$ in K^n is equivalent to solving a system $Ax = b$ of linear equations, where v is the column b of constants, and the u's are the columns of the coefficient matrix A. Such a system may have a unique solution, many solutions,or no solution. The last case – no solution – means that v cannot be written as a linear combination of the u's.

7.1 Linear Combinations of Orthogonal Vectors, Fourier Coefficients

Suppose that $u_1, u_2, \ldots, u_n$ in R^n are n nonzero pairwise orthogonal vectors. This means
(i) $u_i \cdot u_j = 0$ for $i \neq j$ and (ii) $u_i \cdot u_i \neq 0$ for each i.
Then, we can express any v in R^n as a linear combination of $u_1, u_2, \ldots, u_n$.

One method is the usual way to write down the equivalent system of linear equations and reduce the system to echelon form. Other easier method is to take cognisance of the fact the vectors $u_1, u_2, \ldots, u_n$ are mutually orthogonal.

Let us write vector v as follows

$$v = x_1u_1 + x_2u_2 + \cdots + x_nu_n \qquad (3)$$

Take the dot product of Eq. 3 with respect to u_1 to get

$$v \cdot u_1 = x_1u_1 \cdot u_1 + x_2u_2 \cdot u_1 + \cdots + x_nu_n \cdot u_1$$

or $x_1 = \dfrac{v \cdot u_1}{u_1 \cdot u_1}$.

Similarly, taking dot product with respect to u_2, we get

$x_2 = \dfrac{v \cdot u_2}{u_2 \cdot u_2}$ and so on.

We arrive at the following theorem.

Suppose $u_1, u_2, \ldots, u_n$ are nonzero mutually orthogonal vectors in R^n. Then, for any vector v in R^n,

$$v = \frac{v \cdot u_1}{u_1 \cdot u_1}u_1 + \frac{v \cdot u_2}{u_2 \cdot u_2}u_2 + \cdots + \frac{v \cdot u_n}{u_n \cdot u_n}u_n$$

The following scalar k_i (appearing in the theorem) is called the *Fourier coefficient* of v with respect to u_i:

$$k_i = \frac{v \cdot u_i}{u_i \cdot u_i} = \frac{v \cdot u_i}{\|u_i\|^2}$$

It is analogous to a coefficient in the Fourier series of a function.

Example 10 :
Consider the following three vectors in R^3:

$$u_1 = (1, 2, 4), \quad u_2 = (2, -3, 1), \quad u_3 = (2, 1, -1)$$

This vectors are pairwise orthogonal; that is,

$$u_1 \cdot u_2 = 2 - 6 + 4 = 0, \quad u_1 \cdot u_3 = 2 + 2 - 4 = 0,$$

$$u_2 \cdot u_3 = 4 - 3 - 1 = 0$$

Write $v = (7, 16, 6)$ as a linear combination of u_1, u_2, u_3.

Set v as a linear combination of u_1, u_2, u_3 using unknown scalars x, y, z as follows:

$$(7, 16, 6) = x(1, 2, 4) + y(2, -3, 1) + z(2, 1, -1) \qquad (4)$$

Take the dot product of Eq.4 with respect to u_1 to get

$(7, 16, 6) \cdot (1, 2, 4) = x(1, 2, 4) \cdot (1, 2, 4)$ or

$(7+32+24) = (1+4+16)x$ or $63 = 21x$ or $x = 3$

Next take the dot product of Eq.4 with respect to u_2 to obtain

$(7, 16, 6) \cdot (2, -3, 1) = y(2, -3, 1) \cdot (2, -3, 1)$ or

$(14-48+6) = (4+9+1)y$ or $-28 = 14y$ or $y = -2$

Finally, take the dot product of Eq.4 with respect to u_3 to get

$(7, 16, 6) \cdot (2, 1, -1) = z(2, 1, -1) \cdot (2, 1, -1)$ or

$(14 + 16 - 6) = (4 + 1 + 1)z$ or $24 = 6z$ or $z = 4$

Thus $v = 3u_1 - 2u_2 + 4u_3$.

8 Homogeneous Systems of Linear Equations

A system of linear equations is said to be *homogeneous* if all the constant terms are zero ($Ax = 0$). Clearly, such a system has the zero vector $0 = (0, 0, \ldots, 0)$ as a solution, called the *zero* or *trivial* solution. Accordingly, we are usually interested in whether or not the system has a nonzero solution. This we can find out by reducing a homogeneous system $Ax = 0$ to an echelon form. Let r denote the number of equations in echelon form and n denote the number of unknowns. Thus the echelon system has $n-r$ free variables. If $r = n$, then the system has only the zero solution. If $r < n$, then the system has a nonzero solution. Accordingly, if we begin with fewer equations than unknowns, then, in echelon form, $r < n$, and the system has a nonzero solution.

Example 11 :
Determine whether or not each of the following homogeneous systems has a nonzero solution:

(a)

$$\begin{aligned} x + y - z &= 0 \\ 2x - 3y + z &= 0 \\ x - 4y + 2z &= 0 \end{aligned}$$

(b)

$$\begin{aligned} x + y - z &= 0 \\ 2x + 4y - z &= 0 \\ 3x + 2y + 2z &= 0 \end{aligned}$$

(c)

$$\begin{aligned} x_1 + 2x_2 - 3x_3 + 4x_4 &= 0 \\ 2x_1 - 3x_2 + 5x_3 - 7x_4 &= 0 \\ 5x_1 + 6x_2 - 9x_3 + 8x_4 &= 0 \end{aligned}$$

When a system is homogeneous, we represent the system by its coefficient matrix A rather than by its augmented matrix M, since the last column of the augmented matrix M is zero column, and it will remain a zero column during any row-reduction process.

(a) We write the coefficient matrix A and reduce it to echelon form.

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 1 \\ 1 & -4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -5 & 3 \\ 0 & -5 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -5 & 3 \end{bmatrix}$$

(The third row of the second matrix is deleted, since it is a multiple of the second row and will result in a zero row.)
The echelon matrix is written below as linear equations.

$$\begin{aligned} x + y - z &= 0 \\ -5y + 3z &= 0 \end{aligned}$$

The system has a nonzero solution, since there are only two equations in the three unknowns in echelon form. Here z is a free variable. Let us, say, set $z = 5$. Then, by back-substitution, $y = 3$ and $x = 2$. Thus the vector $u = (2, 3, 5)$ is a particular nonzero solution.

(b) We write the coefficient matrix A and reduce it to echelon form.

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 4 & -1 \\ 3 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & -1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 11 \end{bmatrix}$$

The echelon matrix is written below as a system of linear equations.

$$\begin{aligned} x + y - z &= 0 \\ 2y + z &= 0 \\ 11z &= 0 \end{aligned}$$

In echelon form, there are three equations in three unknowns. Thus the system has only the zero solution.

(c) The system must have a nonzero solution, since there are four unknowns but only three equations. (Here we do not need to reduce the system to echelon form.)

8.1 Basis for the General Solution of a Homogeneous System

Let W denote the general solution of a homogeneous system $Ax = 0$. A list of nonzero solution vectors $u_1, u_2, \ldots, u_s$ of the system is said to be a *basis* for W if each solution vector $w \in W$ can be expressed uniquely as a linear combination of the vectors $u_1, u_2, \ldots, u_s$, that is, there exists unique scalars $a_1, a_2, \ldots, a_s$ such that

$$w = a_1u_1 + a_2u_2 + \cdots + a_su_s$$

The number s of such basis vectors is equal to the number of free variables. This number s is called the *dimension* of W, written $\dim W = s$. In case $W = 0$, that is, the system has only the zero solution, we define $\dim W = 0$.

How do we find such a basis? The following theorem answers that question.

Let W be the general solution of a homogeneous system $Ax = 0$, and suppose that the echelon form of the homogeneous system has s free variables. Let $u_1, u_2, \ldots, u_s$ be the solutions obtained by setting one of the free variables equal to 1 (or any nonzero constant) and the remaining free variables equal to 0. Then $\dim W = s$, and the vectors $u_1, u_2, \ldots, u_s$ form a basis of W.

We emphasize that the general solution W may have many bases, and that the above theorem only gives us one such basis.

Example 12 :
Find the dimension and a basis for the general solution W of each of the following homogeneous systems:

(a)

$$\begin{aligned} 2x_1 + 4x_2 - 5x_3 + 3x_4 &= 0 \\ 3x_1 + 6x_2 - 7x_3 + 4x_4 &= 0 \\ 5x_1 + 10x_2 - 11x_3 + 6x_4 &= 0 \end{aligned}$$

(b)

$$\begin{aligned} x - 2y - 3z &= 0 \\ 2x + y + 3z &= 0 \\ 3x - 4y - 2z &= 0 \end{aligned}$$

(a) We reduce the coefficient matrix A to echelon form, obtaining

$$A = \begin{bmatrix} 2 & 4 & -5 & 3 \\ 3 & 6 & -7 & 4 \\ 5 & 10 & -11 & 6 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -5 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -5 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

We write down the corresponding homogeneous system in echelon form:

$$\begin{aligned} 2x_1 + 4x_2 - 5x_3 + 3x_4 &= 0 \\ x_3 - x_4 &= 0 \end{aligned}$$

The system in echelon form has two free variables, x_2 and x_4, so $\dim W = 2$. A basis $[u_1, u_2]$ for W may be obtained as follows:
(1) Set $x_2 = 1$ and $x_4 = 0$. Back-substitution yields $x_3 = 0$, and then $x_1 = -2$. Thus $u_1 = (-2, 1, 0, 0)$.
(2) Set $x_2 = 0$ and $x_4 = 1$. Back-substitution yields $x_3 = 1$, and then $x_1 = 1$. Thus $u_2 = (1, 0, 1, 1)$.

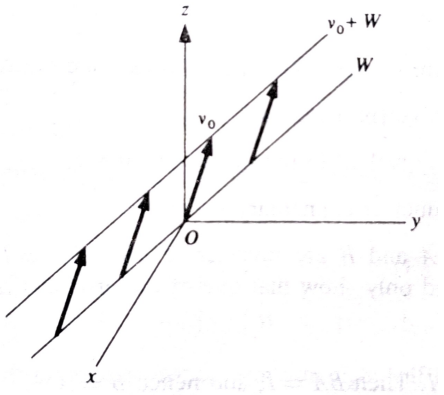

Fig. 1 Geometrical interpretation of relation between solutions U and W.

(b) We reduce the coefficient matrix A to echelon form, obtaining

$$A = \begin{bmatrix} 1 & -2 & -3 \\ 2 & 1 & 3 \\ 3 & -4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -3 \\ 0 & 5 & 9 \\ 0 & 2 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -3 \\ 0 & 5 & 9 \\ 0 & 0 & 17 \end{bmatrix}$$

The corresponding homogeneous system in echelon form is:

$$\begin{aligned} x - 2y - 3z &= 0 \\ 5y + 9z &= 0 \\ 17z &= 0 \end{aligned}$$

There are no free variables (the system is in triangular form). Hence $\dim W = 0$, and W has no basis. Specifically, W consists only of the zero solution, that is, $W = \{0\}$.

8.2 Nonhomogeneous and Associated Homogeneous Systems

The solution U of a nonhomogeneous system $Ax = b$ and the solution W of its *associated homogeneous system* $Ax = 0$ are related. The relationship is given by the following theorem.

Let v_0 be a particular solution of $Ax = b$ and let W be the general solution of $Ax = 0$. Then the following is the general solution of $Ax = b$:

$$U = v_0 + W = \{v_0 + w : w \in W\}$$

That is, $U = v_0 + W$ is obtained by adding v_0 to each element in W.

The solution set W always passes through the origin since zero vector is a solution of the homogeneous system. The above theorem has a geometrical interpretation in R^3. Specifically, suppose W is a line through the origin O. Then, as pictured in Fig. 1, $U = v_0 + W$ is the line parallel to W obtained by adding v_0 to each element of W. Similarly, whever W is a plane through the origin O, then $U = v_0 + W$ is a plane parallel to W.

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